ON THE RIEMANN ZETA-FUNCTION AND THE DIVISOR PROBLEM

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Abstract. Let $\Delta(x)$ denote the error term in the Dirichlet divisor problem, and $E(T)$ the error term in the asymptotic formula for the mean square of $|\zeta(\frac{1}{2}+it)|$. If $E^*(t) = E(t) - 2\pi\Delta^*(t/2\pi)$ with $\Delta^*(x) = -\Delta(x) + 2\Delta(2x) - \frac{1}{2}\Delta(4x)$, then we obtain

$$\int_0^T (E^*(t))^4 \,\mathrm{d}t \ \ll_\varepsilon\ T^{16/9+\varepsilon}.$$

We also show how our method of proof yields the bound

$$\sum_{r=1}^{R}\left(\int_{t_r-G}^{t_r+G} |\zeta(\tfrac{1}{2}+it)|^2 \,\mathrm{d}t\right)^4 \ll_\varepsilon T^{2+\varepsilon}G^{-2} + RG^4T^\varepsilon,$$

where $T^{1/5+\varepsilon} \le G \ll T$, $T < t_1 < \cdots < t_R \le 2T$, $t_{r+1}-t_r \ge 5G\,(r = 1,\ldots,R-1)$.

1. Introduction and statement of results

Let, as usual,

$$\Delta(x) \ = \ \sum_{n\le x} d(n) - x(\log x + 2\gamma - 1) - \tfrac{1}{4}, \tag{1.1}$$

and

$$E(T) \ = \ \int_0^T |\zeta(\tfrac{1}{2}+it)|^2 \,\mathrm{d}t - T\left(\log\left(\frac{T}{2\pi}\right) + 2\gamma - 1\right), \tag{1.2}$$

where $d(n)$ is the number of divisors of $n, \gamma = -\Gamma'(1) = 0.577215\ldots$ is Euler's constant. Thus $\Delta(x)$ denotes the error term in the classical Dirichlet divisor problem, and $E(T)$ is the error term in the mean square formula for $|\zeta(\frac{1}{2}+it)|$.

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An interesting analogy between $d(n)$ and $|\zeta(\frac{1}{2}+it)|^2$ was pointed out by F.V. Atkinson [1] more than sixty years ago. In his famous paper [2], Atkinson continued his research and established an explicit formula for $E(T)$ (see also the author's monographs [7, Chapter 15] and [8, Chapter 2]). The most significant terms in this formula, up to the factor $(-1)^n$, are similar to those in Voronoi's formula (see [7, Chapter 3]) for $\Delta(x)$. More precisely, in [13] M. Jutila showed that $E(T)$ should be actually compared to $2\pi\Delta^*(T/(2\pi))$, where

$$\Delta^*(x) \;:=\; -\Delta(x) + 2\Delta(2x) - \tfrac{1}{2}\Delta(4x). \tag{1.3}$$

Then the arithmetic interpretation of $\Delta^*(x)$ (see T. Meurman [16]) is

$$\tfrac{1}{2}\sum_{n\le 4x}(-1)^n d(n) \;=\; x(\log x + 2\gamma - 1) + \Delta^*(x). \tag{1.4}$$

We have the explicit, truncated formula (see e.g., [7] or [18])

$$\Delta(x) = \frac{1}{\pi\sqrt{2}}x^{\frac14}\sum_{n\le N} d(n)n^{-\frac34}\cos(4\pi\sqrt{nx}-\tfrac14\pi) + O_\varepsilon(x^{\frac12+\varepsilon}N^{-\frac12}) \quad (2\le N\ll x). \tag{1.5}$$

One also has (see [7, eq. (15.68)]), for $2\le N\ll x$,

$$\Delta^*(x) = \frac{1}{\pi\sqrt{2}}x^{\frac14}\sum_{n\le N}(-1)^n d(n)n^{-\frac34}\cos(4\pi\sqrt{nx}-\tfrac14\pi) + O_\varepsilon(x^{\frac12+\varepsilon}N^{-\frac12}), \tag{1.6}$$

which is completely analogous to (1.5).

M. Jutila, in his works [13] and [14], investigated both the local and global behaviour of

$$E^*(t) \;:=\; E(t) - 2\pi\Delta^*\Big(\frac{t}{2\pi}\Big).$$

He proved the mean square bound

$$\int_{T-H}^{T+H}(E^*(t))^2\,\mathrm{d}t \ll_\varepsilon HT^{1/3}\log^3 T + T^{1+\varepsilon} \quad (1\ll H\le T), \tag{1.7}$$

which in particular yields

$$\int_0^T (E^*(t))^2\,\mathrm{d}t \ll T^{4/3}\log^3 T. \tag{1.8}$$

Here and later ε denotes positive constants which are arbitrarily small, but are not necessarily the same at each occurrence. The bound (1.8) shows that, on the

average, $E^*(t)$ is of the order $\ll t^{1/6}\log^{3/2} t$, while both $E(x)$ and $\Delta(x)$ are of the order $\asymp x^{1/4}$. This follows from the mean square formulas (see e.g., [8])

$$\int_0^T \Delta^2(x)\,\mathrm{d}x = (6\pi^2)^{-1}\sum_{n=1}^\infty d^2(n)n^{-3/2}T^{3/2} + O(T\log^4 T), \tag{1.9}$$

and

$$\int_0^T E^2(x)\,\mathrm{d}x = \tfrac{2}{3}(2\pi)^{-1/2}\sum_{n=1}^\infty d^2(n)n^{-3/2}T^{3/2} + O(T\log^4 T). \tag{1.10}$$

The mean square formulas (1.9) and (1.10) also imply that the inequalities $\alpha < 1/4$ and $\beta < 1/4$ cannot hold, where α and β are, respectively, the infima of the numbers a and b for which the bounds

$$\Delta(x) \ll x^a, \qquad E(x) \ll x^b \tag{1.11}$$

hold. For upper bounds on α, β see e.g., M.N. Huxley [5]. Classical conjectures are that $\alpha = \beta = 1/4$ holds, although this is notoriously difficult to prove. M. Jutila [13] succeeded in showing the conditional estimates: if the conjectural $\alpha = 1/4$ holds, then this implies that $\beta \le 3/10$. Conversely, $\beta = 1/4$ implies that $\Delta^*(x) \ll_\varepsilon x^{\theta+\varepsilon}$ holds with $\theta \le 3/10$. Although one expects the maximal orders of $\Delta(x)$ and $\Delta^*(x)$ to be approximately of the same order of magnitude, this does seems difficult to establish.

In what concerns the formulas involving higher moments of $\Delta(x)$ and $E(t)$, we refer the reader to the author's works [6], [7] and [10] and D.R. Heath-Brown [4]. In particular, note that [10] contains proofs of

$$\begin{aligned} \int_0^T E^3(t)\,\mathrm{d}t &= 16\pi^4\int_0^{\frac{T}{2\pi}} (\Delta^*(t))^3\,\mathrm{d}t + O(T^{5/3}\log^{3/2} T),\\ \int_0^T E^4(t)\,\mathrm{d}t &= 32\pi^5\int_0^{\frac{T}{2\pi}} (\Delta^*(t))^4\,\mathrm{d}t + O(T^{23/12}\log^{3/2} T). \end{aligned} \tag{1.12}$$

In a recent work by P. Sargos and the author [12], the asymptotic formulas of K.-M. Tsang [19] for the cube and the fourth moment of $\Delta(x)$ were sharpened to

$$\int_1^X \Delta^3(x)\,\mathrm{d}x = BX^{7/4} + O_\varepsilon(X^{\beta+\varepsilon}) \qquad (B > 0) \tag{1.13}$$

and

$$\int_1^X \Delta^4(x)\,\mathrm{d}x = CX^2 + O_\varepsilon(X^{\gamma+\varepsilon}) \qquad (C>0) \tag{1.14}$$

with $\beta = \frac{7}{5}, \gamma = \frac{23}{12}$. This improves on the values $\beta = \frac{47}{28}, \gamma = \frac{45}{23}$, obtained in [19]. Moreover, (1.13) and (1.14) remain valid if $\Delta(x)$ is replaced by $\Delta^*(x)$, since their proofs used nothing more besides (1.5) and the bound $d(n) \ll_\varepsilon n^\varepsilon$. Hence from (1.12) and the analogues of (1.13)–(1.14) for $\Delta^*(x)$, we infer then that

$$\begin{aligned} \int_0^T E^3(t)\,\mathrm{d}t \;&=\; B_1T^{7/4} + O(T^{5/3}\log^{3/2}T) \quad (B_1>0), \\ \int_0^T E^4(t)\,\mathrm{d}t \;&=\; C_1T^2 + O_\varepsilon(T^{23/12+\varepsilon}) \quad (C_1>0). \end{aligned} \tag{1.15}$$

The main aim of this paper is to provide an estimate for the upper bound of the fourth moment of $E^*(t)$, which is the first non-trivial upper bound for a higher moment of $E^*(t)$. The result is the following

THEOREM 1. *We have*

$$\int_0^T (E^*(t))^4\,\mathrm{d}t \;\ll_\varepsilon\; T^{16/9+\varepsilon}. \tag{1.16}$$

Note that the bounds (1.8) and (1.16) do not seem to imply each other. For the proof of (1.16) we shall need several lemmas, which will be given in Section 2. The proof of Theorem 1 will be given in Section 3. Finally, in Section 4, it will be shown how the method of proof of Theorem 1 can give a proof of

THEOREM 2. *Let* $T^{1/5+\varepsilon} \le G \ll T$, $T < t_1 < \cdots < t_R \le 2T$, $t_{r+1} - t_r \ge 5G\,(r = 1, \cdots, R-1)$. *Then*

$$\sum_{r=1}^R \left(\int_{t_r-G}^{t_r+G} |\zeta(\tfrac12+it)|^2\,\mathrm{d}t\right)^4 \ll_\varepsilon T^{2+\varepsilon}G^{-2} + RG^4T^\varepsilon. \tag{1.17}$$

The bound in (1.17) easily gives the well-known bound (see Section 4)

$$\int_0^T |\zeta(\tfrac12+it)|^{12}\,\mathrm{d}t \ll_\varepsilon T^{2+\varepsilon}, \tag{1.18}$$

due to D.R. Heath-Brown [3] (who had $\log^{17} T$ instead of the T^ε-factor). It is still essentially the sharpest result concerning high moments of $|\zeta(\frac12+it)|$. General sums of zeta-integrals over short intervals, analogous to the one appearing in (1.17), were treated by the author in [9].

2. The necessary lemmas

LEMMA 1 (O. Robert–P. Sargos [17]). *Let $k \ge 2$ be a fixed integer and $\delta > 0$ be given. Then the number of integers n_1, n_2, n_3, n_4 such that $N < n_1, n_2, n_3, n_4 \le 2N$ and*

$$|n_1^{1/k} + n_2^{1/k} - n_3^{1/k} - n_4^{1/k}| < \delta N^{1/k}$$

is, for any given $\varepsilon > 0$,

$$\ll_\varepsilon N^\varepsilon (N^4\delta + N^2). \tag{2.1}$$

LEMMA 2. *Let $1 \ll G \ll T/\log T$. Then we have*

$$E(T) \le \frac{2}{\sqrt{\pi}G}\int_0^\infty E(T+u)\,\mathrm{e}^{-u^2/G^2}\,\mathrm{d}u + O(G\log T), \tag{2.2}$$

and

$$E(T) \ge \frac{2}{\sqrt{\pi}G}\int_0^\infty E(T-u)\,\mathrm{e}^{-u^2/G^2}\,\mathrm{d}u + O(G\log T). \tag{2.3}$$

Proof of Lemma 2. The proofs of (2.2) and (2.3) are analogous, so only the former will be treated in detail. From (1.2) we have, for $0 \le u \ll T$,

$$0 \le \int_T^{T+u} |\zeta(\tfrac12 + it)|^2\,\mathrm{d}t = (T+u)\Bigl(\log\bigl(\frac{T+u}{2\pi}\bigr) + 2\gamma - 1\Bigr) - T\Bigl(\log\bigl(\frac{T}{2\pi}\bigr) + 2\gamma - 1\Bigr) + E(T+u) - E(T).$$

This gives

$$E(T) \le E(T+u) + O(u\log T),$$

hence

$$\int_0^{G\log T} E(T)\,\mathrm{e}^{-u^2/G^2}\,\mathrm{d}u \le \int_0^{G\log T} (E(T+u) + O(u\log T))\,\mathrm{e}^{-u^2/G^2}\,\mathrm{d}u.$$

The proof of (2.2) is completed when we extend the integration to $[0,\infty)$ making a small error, and recall that $\int_0^\infty \mathrm{e}^{-u^2/G^2}\,\mathrm{d}u = \frac12\sqrt{\pi}G$, $\int_0^\infty u\mathrm{e}^{-u^2/G^2}\,\mathrm{d}u = \frac12 G$.

LEMMA 3. *Let $1 \ll G \ll T$. Then we have*

$$\Delta^*\bigl(\frac{T}{2\pi}\bigr) = \frac{2}{\sqrt{\pi}G}\int_0^\infty \Delta^*\bigl(\frac{T}{2\pi} \pm \frac{u}{2\pi}\bigr)\,\mathrm{e}^{-u^2/G^2}\,\mathrm{d}u + O_\varepsilon(GT^\varepsilon). \tag{2.4}$$

Proof of Lemma 3. Both the cases of the + and − sign in (2.4) are treated analogously. For example, we have

$$
\begin{aligned}
&\tfrac12\sqrt{\pi}G\Delta^*\bigl(\frac{T}{2\pi}\bigr) - \int_0^\infty \Delta^*\bigl(\frac{T}{2\pi}+\frac{u}{2\pi}\bigr)\,\mathrm{e}^{-u^2/G^2}\,\mathrm{d}u\\
&= \int_0^\infty \Bigl(\Delta^*(T) - \Delta^*\bigl(\frac{T}{2\pi}+\frac{u}{2\pi}\bigr)\Bigr)\,\mathrm{e}^{-u^2/G^2}\,\mathrm{d}u\\
&= \int_0^{G\log T} \Biggl(\Delta^*\bigl(\frac{T}{2\pi}\bigr) - \Delta^*\bigl(\frac{T}{2\pi}+\frac{u}{2\pi}\bigr)\Biggr)\,\mathrm{e}^{-u^2/G^2}\,\mathrm{d}u + O(1)\\
&\ll \int_0^{G\log T}\Bigl\{\Bigl|\sum_{\frac{2}{\pi}T\le n\le\frac{2}{\pi}(T+u)}(-1)^n d(n)\Bigr| + O((1+|u|)\log T)\Bigr\}\,\mathrm{d}u \ll_\varepsilon G^2T^\varepsilon,
\end{aligned}
$$

where we used (1.4) and $d(n)\ll_\varepsilon n^\varepsilon$. This establishes (2.4).

The next lemma is F.V. Atkinson's classical explicit formula for $E(T)$ (see [2], [7] or [8]).

LEMMA 4. *Let $0 < A < A'$ be any two fixed constants such that $AT < N < A'T$, and let $N' = N'(T) = T/(2\pi) + N/2 - (N^2/4 + NT/(2\pi))^{1/2}$. Then*

$$E(T) = \Sigma_1(T) + \Sigma_2(T) + O(\log^2 T), \tag{2.5}$$

where

$$\Sigma_1(T) = 2^{1/2}(T/(2\pi))^{1/4}\sum_{n\le N}(-1)^n d(n)n^{-3/4}e(T,n)\cos(f(T,n)), \tag{2.6}$$

$$\Sigma_2(T) = -2\sum_{n\le N'} d(n)n^{-1/2}(\log T/(2\pi n))^{-1}\cos(T\log T/(2\pi n) - T + \pi/4), \tag{2.7}$$

with

$$
\begin{aligned}
&f(T,n) = 2T\operatorname{arsinh}\bigl(\sqrt{\pi n/(2T)}\bigr) + \sqrt{2\pi nT+\pi^2n^2} - \pi/4\\
&= -\tfrac14\pi + 2\sqrt{2\pi nT} + \tfrac16\sqrt{2\pi^3}n^{3/2}T^{-1/2} + a_5n^{5/2}T^{-3/2} + a_7n^{7/2}T^{-5/2} + \dots,
\end{aligned}
\tag{2.8}
$$

$$
\begin{aligned}
e(T,n) &= (1+\pi n/(2T))^{-1/4}\Bigl\{(2T/\pi n)^{1/2}\operatorname{arsinh}\bigl(\sqrt{\pi n/(2T)}\bigr)\Bigr\}^{-1}\\
&= 1 + O(n/T) \qquad (1\le n<T),
\end{aligned}
\tag{2.9}
$$

and $\operatorname{arsinh} x = \log(x + \sqrt{1+x^2}\,)$.

LEMMA 5 (M. Jutila [13]). *For* $A \in \mathbb{R}$ *we have*

(2.10)
$$\cos\left(\sqrt{8\pi nT} + \tfrac{1}{6}\sqrt{2\pi^3}n^{3/2}T^{-1/2} + A\right) = \int_{-\infty}^{\infty} \alpha(u)\cos(\sqrt{8\pi n}(\sqrt{T}+u)+A)\,\mathrm{d}u,$$

where $\alpha(u) \ll T^{1/6}$ *for* $u \ne 0$,

$$\alpha(u) \ll T^{1/6}\exp(-bT^{1/4}|u|^{3/2}) \tag{2.11}$$

for $u < 0$, *and*

(2.12)
$$\alpha(u) = T^{1/8}u^{-1/4}\left(d\exp(ibT^{1/4}u^{3/2}) + \bar{d}\exp(-ibT^{1/4}u^{3/2})\right) + O(T^{-1/8}u^{-7/4})$$

for $u \ge T^{-1/6}$ *and some constants* b (>0) *and* d.

3. The proof of Theorem 1

We shall prove that

$$\int_T^{2T} (E^*(t))^4\,\mathrm{d}t \;\ll_\varepsilon\; T^{16/9+\varepsilon}, \tag{3.1}$$

which easily implies (1.16) on replacing T by $T/2, T/2^2, \ldots$ etc. and summing all the results. Henceforth we assume that $T \le t \le 2T$, $T^\varepsilon \le G \ll T^{5/12}$, and we begin by evaluating the integrals

$$\int_0^{\infty} E(t \pm u)\mathrm{e}^{-u^2/G^2}\,\mathrm{d}u \tag{3.2}$$

which appear in Lemma 2 (with t replacing T), truncating them at $u = G\log T$ with a negligible error. A similar procedure was effected by D.R. Heath-Brown [4] and by the author [7, Chapter 7], where the details of analogous estimations may be found. It transpires that the contribution of $\Sigma_2(T)$ (see (2.7)) in Atkinson's formula, as well as the contribution of n in $\Sigma_1(T)$ which satisfy $n > TG^{-2}\log T$ will be $\ll G\log T$, if we take in Lemma 4 $N = T$ for $E(t)$ when $T \le t \le 2T$. What remains clearly corresponds to the truncated formula (1.6) for $\Delta^*(x)$ with $N = TG^{-2}\log T$, or equivalently

$$G \;=\; \sqrt{\frac{T}{N}\log T}. \tag{3.3}$$

We combine now (2.2) with (2.4) with the + sign (when $E(T) \ge 0$) or (2.3) with (2.4) with the − sign (when $E(T) \le 0$), to obtain by the Cauchy-Schwarz inequality

$$(E^*(t))^2 \ll_\varepsilon G^{-1} \int_{-G\log T}^{G\log T} \mathrm{e}^{-u^2/G^2} (E^*(t+u))^2 \,\mathrm{d}u + G^2T^\varepsilon, \tag{3.4}$$

provided that $T \le t \le 2T, T^\varepsilon \ll G \ll T^{5/12}$. Keeping in mind the preceding discussion we thus have (replacing $(t+u)^{1/4}$ with $t^{1/4}$ by Taylor's formula, with the error absorbed by the last term in (3.5)) by using (1.6), (2.5), (3.3) and (3.4),

$$(E^*(t))^2 \ll_\varepsilon G^{-1} \int_{-G\log T}^{G\log T} \mathrm{e}^{-u^2/G^2} (\Sigma_3^2(X;u) + \Sigma_4^2(X,N;u) + \Sigma_5^2(X,N;u)) \,\mathrm{d}u + T^{1+\varepsilon}N^{-1}, \tag{3.5}$$

where we set

$$\Sigma_3(X;u) := t^{1/4} \times \sum_{n\le X} (-1)^n d(n) n^{-3/4} \Big\{ e(t+u,n)\cos(f(t+u,n)) - \cos(\sqrt{8\pi n(t+u)} - \pi/4) \Big\}, \tag{3.6}$$

$$\begin{aligned} \Sigma_4(X,N;u) &:= t^{1/4} \sum_{X<n\le N} (-1)^n d(n) n^{-3/4} e(t+u,n)\cos(f(t+u,n)), \\ \Sigma_5(X,N;u) &:= t^{1/4} \sum_{X<n\le N} (-1)^n d(n) n^{-3/4} \cos(\sqrt{8\pi n(t+u)} - \pi/4), \end{aligned} \tag{3.7}$$

where we suppose that ($N = N(T)$ is the analogue of N in (1.5) and (1.6) (cf. (3.4)), and not of N in Lemma 4)

$$T^\varepsilon \le X < T^{1/3}, \ \max(X, T^{1/6}\log T) < N \ll T^{11/17}. \tag{3.8}$$

Here $X = X(T)$ is a parameter which allows one (by using (2.8)) to replace, in $\Sigma_3(X;u)$, $\cos(f(t+u,n))$ by

$$(1 + cn^{3/2}(t+u)^{-1/2})\cos(\sqrt{8\pi n(t+u)} - \pi/4)$$

plus terms of a lower order of magnitude. Note that, for $n \le X$ $(< T^{1/3})$, we may also replace $e(t+u,n)$ in (2.6) by 1 with the error absorbed by the last

term in (3.5). The conditions imposed in (3.8) imply that G (see (3.4)) satisfies $G \ll T^{5/12}$. Hence instead of $\Sigma_3(X;u)$ in (3.5), we may estimate

$$\Sigma_6(X;u) := \sum_{n\le X} t^{-1/4}(-1)^n d(n) n^{3/4}\cos(\sqrt{8\pi n(t+u)}-\pi/4), \tag{3.9}$$

which has the advantage because the cosine contains $\sqrt{8\pi n(t+u)}-\pi/4$ instead of the more complicated function $f(t+u,n)$. Thus with the aid of (3.5)–(3.9) we see that the left-hand side of (3.1) is majorized by the maximum taken over $|u| \le G\log T$ times

(3.10)

$$\begin{aligned}&\int_T^{2T}(E^*(t))^2(\Sigma_4^2(X,N;u)+\Sigma_5^2(X,N;u)+\Sigma_6^2(X;u)+T^{1+\varepsilon}N^{-1})\,\mathrm{d}t\\ &\ll_\varepsilon \left\{\int_T^{2T}(E^*(t))^4\,\mathrm{d}t\int_T^{2T}\Bigl(\Sigma_4^4(X,N;u)+\Sigma_5^4(X,N;u)+\Sigma_6^4(X;u)\Bigr)\,\mathrm{d}t\right\}^{1/2}\\ &+T^{7/3+\varepsilon}N^{-1},\end{aligned}$$

where we used the Cauchy-Schwarz inequality for integrals and (1.8). Thus from (3.10) we have the key bound

(3.11)

$$\begin{aligned}\int_T^{2T}(E^*(t))^4\,\mathrm{d}t \ll_\varepsilon \max_{|u|\le G\log T}\int_T^{2T}\Bigl(\Sigma_4^4(X,N;u)+\Sigma_5^4(X,N;u)+\Sigma_6^4(X;u)\Bigr)\,\mathrm{d}t\\ +T^{7/3+\varepsilon}N^{-1}.\end{aligned}$$

To evaluate the integrals on the right-hand side of (3.11) we note first that

$$\int_T^{2T}\Bigl(\Sigma_4^4(X,N;u)+\dots\Bigr)\,\mathrm{d}t \le \int_{T/2}^{5T/2}\varphi(t)\Bigl(\Sigma_4^4(X,N;u)+\dots\Bigr)\,\mathrm{d}t, \tag{3.12}$$

where $\varphi(t)$ is a smooth, nonnegative function supported in $[T/2,\,5T/2]$, such that $\varphi(t)=1$ when $T\le t\le 2T$. The integrals of $\sum_4^4(X,N;u), \sum_5^4(X,N;u)$ and $\sum_6^4(X;u)$ are all estimated analogously. The sums over n are divided into $\ll \log T$ subsums of the form $\sum_{K<n\le K'\le 2K}$, the cosines are written as exponentials, and the fourth power is written as a quadruple sum over the integer variables m,n,k,l. Then we perform a large number of integrations by parts to deduce that the contribution of those m,n,k,l for which $|\Delta|\ge T^{\varepsilon-1/2}$ is negligible (i.e., $\ll T^{-A}$ for any fixed $A>0$), where

$$\Delta := \sqrt{8\pi}(\sqrt{m}+\sqrt{n}-\sqrt{k}-\sqrt{l}\,). \tag{3.13}$$

Therefore, in the case of $\Sigma_5(X,N;u)$, there remains the estimate

(3.14)
$$\begin{aligned}
&\int_T^{2T}\Sigma_5^4(X,N;u)\,\mathrm{d}t\\
&\ll_\varepsilon 1+T^{1+\varepsilon}\max_{|u|\le G\log T}\sup_{X\le K\le N}\int_{T/2}^{5T/2}\varphi(t)\times\\
&\Bigl|\sideset{}{^*}\sum_{K<m,n,k,l\le K'\le 2K}(-1)^{m+n+k+l}d(m)d(n)d(k)d(l)(mnkl)^{-3/4}\exp(i\Delta\sqrt{t+u})\Bigr|\,\mathrm{d}t,
\end{aligned}$$

where $\sum^*$ means that $|\Delta|\le T^{\varepsilon-1/2}$ holds. Now we use Lemma 1 (with $k=2$, $\delta\asymp K^{-1/2}|\Delta|$), estimating the integral on the right-hand side of (3.14) trivially. We obtain that the left-hand side of (3.14) is

$$\begin{aligned}
&\ll_\varepsilon T^{1+\varepsilon}\max_{X\le K\le N,|\Delta|\le T^{\varepsilon-1/2}}K^{-3}T(K^4K^{-1/2}|\Delta|+K^2)\\
&\ll_\varepsilon T^\varepsilon(T^2N^{1/2}T^{-1/2}+T^2X^{-1})\\
&\ll_\varepsilon T^{3/2+\varepsilon}N^{1/2}+T^{2+\varepsilon}X^{-1}.
\end{aligned}\tag{3.15}$$

Proceeding analogously as in (3.15), we obtain that

$$\begin{aligned}
\int_T^{2T}\Sigma_6^4(X;u)\,\mathrm{d}t &\ll_\varepsilon T^{1+\varepsilon}\max_{1\le K\le X,|\Delta|\le T^{\varepsilon-1/2}}T^{-1}K^3(K^4K^{-1/2}|\Delta|+K^2)\\
&\ll_\varepsilon T^\varepsilon(T^{-1/2}X^{13/2}+X^5),
\end{aligned}\tag{3.16}$$

since instead of $(mnkl)^{-3/4}$ in (3.14) now we shall have $(mnkl)^{3/4}t^{-1}$ (see (3.9)).

The estimation of $\Sigma_4(X,N;u)$ (see (3.7)) presents a technical problem, since the cosines contain the function $f(t,n)$, and Lemma 1 cannot be applied directly. First we note that, by using (2.8), we can expand the exponential in power series to get rid of the terms $a_5n^{5/2}t^{-3/2}+\ldots$. In this process the main term will be 1, and the error terms will make a contribution which will be (for shortness we set

$a = \sqrt{8\pi}, b = \frac{1}{6}\sqrt{2\pi^3}$ and $\tau = t + u$)

$$\ll_\varepsilon \max_{|u|\le G\log T} \sup_{X\le K\le N} T\int_{T/2}^{5T/2} \varphi(t)\Big|\sum_{K<n\le 2K}(-1)^n d(n)n^{7/4}\tau^{-3/2}\times$$
$$\times \exp\Big(ia(n\tau)^{1/2} + ib(n^3/\tau)^{1/2}\Big)\Big|^4 \,\mathrm{d}t$$
$$\ll_\varepsilon \max_{|u|\le G\log T} \sup_{X\le K\le N} T^{\varepsilon-5}K^{9/2}\int_{T/2}^{5T/2} \varphi(t)\Big|\sum_{K<n\le 2K}(-1)^n d(n)n^{7/4}\times$$
$$\times \exp\Big(ia(n\tau)^{1/2} + ib(n^3/\tau)^{1/2}\Big)\Big|^2 \,\mathrm{d}t$$
$$\ll_\varepsilon \max_{|u|\le G\log T} \sup_{X\le K\le N} T^{\varepsilon-5}K^{9/2}(T\sum_{K<n\le 2K} n^{7/2}$$
$$+ T^{1/2}\sum_{K<m\ne n\le 2K}(mn)^{7/4}|\sqrt{m}-\sqrt{n}|^{-1})$$
$$\ll_\varepsilon \max_{|u|\le G\log T} \sup_{X\le K\le N} T^{\varepsilon-5}K^{9/2}(TK^{9/2} + +T^{1/2}K^4\sum_{K<m\ne n\le 2K}|m-n|^{-1})$$
$$\ll_\varepsilon \max_{X\le K\le N} T^{\varepsilon-5}K^{9/2}TK^{9/2} \ll_\varepsilon T^{\varepsilon-4}N^9 \ll_\varepsilon T^{3/2+\varepsilon}N^{1/2}$$

for $N \ll T^{11/17}$, which is implied by (3.8). Thus we are left with

$$\cos\left(\sqrt{8\pi n\tau} + \tfrac{1}{6}\sqrt{2\pi^3}n^{3/2}\tau^{-1/2} - \tfrac{1}{4}\pi\right)$$

in $\Sigma_4(X, N; u)$, and we can apply Lemma 5. With $\alpha(v)$ given by (2.12) we have (3.17)

$$\cos\left(\sqrt{8\pi n\tau} + \tfrac{1}{6}\sqrt{2\pi^3}n^{3/2}\tau^{-1/2} - A\right) = O(T^{-10}) +$$
$$\int_{-u_0}^{u_1}\alpha(v)\cos(\sqrt{8\pi n}(\sqrt{\tau}+v) - A)\,\mathrm{d}v + \int_{u_1}^{\infty}\alpha(v)\cos(\sqrt{8\pi n}(\sqrt{\tau}+v) - A)\,\mathrm{d}v,$$

where we set

$$u_0 = T^{-1/6}\log T,\ u_1 = CKT^{-1/2}, \tag{3.18}$$

and $C > 0$ is a large constant.

We proceed now as in the case of $\Sigma_5(X, N; u)$. We write the cosines as exponentials in the quadruple sum over m, n, k, l. Again, after we first perform

a large number of integrations by parts over t, only the portion of the sum for which $|\Delta| \le T^{\varepsilon-1/2}$ will remain, where Δ is given by (3.13). In the remaining sum we use (3.17) (once with $A = \frac14\pi$ and once with $A = \frac34\pi$), noting that $e^{iz} = \cos z + i\cos(z - \frac12\pi)$. We remark that, for $|v| \le u_0$, we can use the crude estimate $\alpha(u) \ll T^{1/6}$, hence for this portion the estimation will be quite analogous to the preceding case. Next we note that

$$\int_{u_0}^{u_1} \tau^{1/8} v^{-1/4} \exp(ib\tau^{1/4}v^{3/2} \pm \sqrt{8\pi n v})\,dv \ll \log T \quad (\tau = t+u,\ |u| \le G\log T),$$

writing the integral as a sum of $\ll \log T$ integrals over $[U, U']$ with $u_0 \le U < U' \le 2U \ll u_1$, and applying the second derivative test to each of these integrals. We also remark that the contribution of the O-term in (2.12) will be, by trivial estimation,

$$\int_{u_0}^{\infty} T^{-1/8}u^{-7/4}\,du \ll T^{-1/8}u_0^{-3/4} \ll 1$$

if we suppose that (3.18) is satisfied. It remains yet to deal with the integral with $v > u_1$ in (3.17), when we note that

$$\frac{\partial}{\partial v}\left(b\tau^{1/4}v^{3/2} \pm \sqrt{8\pi n v}\right) \gg T^{1/4}v^{1/2} \quad (v > u_1),$$

provided that C in (3.18) is sufficiently large. Hence by the first derivative test

$$\begin{aligned}
&\int_{u_1}^{\infty} \alpha(v)\cos(\sqrt{8\pi n}(\sqrt{\tau}+v) - \tfrac14\pi)\,dv\\
&\ll 1 + T^{1/8}u_1^{-1/4}T^{-1/4}u_1^{-1/2}\\
&\ll 1 + T^{1/4}K^{-3/4} \ll 1 + T^{1/4}X^{-3/4} \ll 1,
\end{aligned}$$

since $K \gg X \gg T^{1/3}$. Thus the contribution of the integrals on the right-hand side of (3.17) is $\ll \log T$.

Then we can proceed with the estimation as in the case of $\Sigma_5(X, N; u)$ to obtain

$$\int_T^{2T} \Sigma_4^4(X, N; u)\,dt \ll_\varepsilon T^{3/2+\varepsilon}N^{1/2} + T^{2+\varepsilon}X^{-1}.$$

Gathering together all the bounds, we see that the integral in (3.1) is

$$\ll_\varepsilon T^{\varepsilon}\Big(T^{3/2}N^{1/2} + T^2X^{-1} + T^{-1/2}X^{13/2} + X^5 + T^{7/3+\varepsilon}N^{-1}\Big), \tag{3.19}$$

provided that (3.8) holds. Finally we choose

$$X = T^{1/3-\varepsilon}, \quad N = T^{5/9},$$

so that (3.8) is fulfilled. The above terms are then $\ll_\varepsilon T^{16/9+\varepsilon}$, and the proof of Theorem 1 is complete. The limit of the method is the bound $\ll T^2X^{-1} \ll T^{5/3}$, which would yield the exponent $5/3+\varepsilon$ in (1.16). The true order of the integral in (1.16), and in general the order of the k-th moment of $E^*(t)$, is elusive. This comes from the definition $E^*(t) = E(t) - 2\pi\Delta^*(t/(2\pi))$, which makes it difficult to see how much the oscillations of the functions E and Δ^* cancel each other.

4. The proof of Theorem 2

We shall show now how the method of proof of our Theorem 1 may be used to yield Theorem 2. Our starting point is an expression for the integral

$$\int_{t_r-2G}^{t_r+2G} \varphi_r(t)|\zeta(\tfrac12+it)|^2\,\mathrm{d}t, \tag{4.1}$$

where t_r is as in the formulation of Theorem 2, and $\varphi_r \in C^\infty$ is a non-negative function supported in $[t_r-2G,\, t_r+2G]$ that equals unity in $[t_r-G,\, t_r+G]$. The integral in (4.1) majorizes the integral

$$\int_{t_r-G}^{t_r+G} |\zeta(\tfrac12+it)|^2\,\mathrm{d}t, \tag{4.2}$$

which is of great importance in zeta-function theory (see K. Matsumoto [15] for an extensive account on mean square theory involving $\zeta(s)$). One can treat the integral in (4.1) by any of the following methods.

a) Using exponential averaging (or some other smoothing like φ_r above), namely the Gaussian weight $\exp(-\frac12x^2)$, in connection with the function $E(T)$, in view of F.V. Atkinson's well-known explicit formula (cf. Lemma 4). This is the approach employed originally by D.R. Heath-Brown [3].

b) One can use the Voronoi summation formula (e.g., see [8, Chapter 3]) for the explicit expression (approximate functional equation) for $|\zeta(\frac12+it)|^2 = \chi^{-1}(\frac12+it)\zeta^2(\frac12+it)$, where $\zeta(s)=\chi(s)\zeta(1-s)$, namely

$$\chi(s) = 2^s\pi^{s-1}\sin(\tfrac12\pi s)\Gamma(1-s).$$

Voronoi's formula is present indirectly in Atkinson's formula, so that this approach is more direct. The effect of the smoothing function φ_r in (4.2) is to shorten the sum approximating $|\zeta|^2$ to the range $\frac{T}{2\pi}(1-G^{-1}T^\varepsilon) \le n \le \frac{T}{2\pi}$ $(T=t_r)$. After

this no integration is needed, and proceeding as in [7, Chapters 7-8] one obtains that the integral in (4.2) equals $O_\varepsilon(GT^\varepsilon)$ plus a multiple of

$$\int_{t_r-2G}^{t_r+2G}\varphi_r(t)\sum_{k\le T^{1+\varepsilon}G^{-2}}(-1)^k d(k)k^{-1/2}\left(\frac{1}{4}+\frac{t}{2\pi k}\right)^{-1/4}\sin f(t,k)\,\mathrm{d}t, \tag{4.3}$$

where $f(t,k)$ is given by (2.8).

c) Instead of the Voronoi summation formula one can use the (simpler) Poisson summation formula, namely

$$\sum_{n=1}^\infty f(n) = \int_0^\infty f(x)\,\mathrm{d}x + 2\sum_{n=1}^\infty\int_0^\infty f(x)\cos(2\pi nx)\,\mathrm{d}x,$$

provided that $f(x)$ is smooth and compactly supported in $(0,\infty)$. In [11] a sketch of this approach is given.

We begin now the derivation of (1.17), simplifying first in (4.3) the factor $(1/4+t/(2\pi k))^{-1/4}$ by Taylor's formula, and then raising the expression in (4.3) to the fourth power, using Hölder's inequality for integrals. It follows that the sum in (1.17) is bounded by

(4.4)

$$\begin{aligned} &RG^4T^\varepsilon + T^{-1}G^3\sum_{r=1}^R\int_{t_r-2G}^{t_r+2G}\varphi_r(t)\Big|\sum_{k\le T^{1+\varepsilon}G^{-2}}(-1)^k d(k)k^{-1/4}\sin f(t,k)\Big|^4\,\mathrm{d}t\\ &\ll_\varepsilon RG^4T^\varepsilon + T^{-1}G^3\int_{T/2}^{5T/2}\varphi(t)\Big|\sum_{k\le T^{1+\varepsilon}G^{-2}}(-1)^k d(k)k^{-1/4}\sin f(t,k)\Big|^4\,\mathrm{d}t, \end{aligned}$$

where $\varphi(t)$ is a non-negative, smooth function supported in $[T/2, 5T/2]$ such that $\varphi(t)=1$ for $T\le t\le 2T$, hence $\varphi^{(m)}(t)\ll_m T^{-m}$. Therefore it suffices to bound the expression

$$I_K := \int_{T/2}^{5T/2}\varphi(t)\Big|\sum_{K<k\le K'\le 2K}(-1)^k d(k)k^{-1/4}\mathrm{e}^{if(t,k)}\Big|^4\,\mathrm{d}t, \tag{4.5}$$

where $T^{1/3}\le K\ll T^{1+\varepsilon}G^{-2}$, $T^{1/5+\varepsilon}\le G\le T^{1/3}$. Namely for $K\le T^{1/3}$ the contribution is trivially $\ll RG^4T^\varepsilon$, and the same holds (e.g., see [7, Theorem 7.3]) for the values $G\ge T^{1/3}$. Recall that

$$f(t,k) = -\tfrac14\pi + 2\sqrt{2\pi kt} + \tfrac16\sqrt{2\pi^3}k^{3/2}t^{-1/2} + a_5k^{5/2}t^{-3/2} + a_7k^{7/2}t^{-5/2} + \dots\,,$$

and note that we have $k^{5/2}t^{-3/2} \ll T^{1+\varepsilon}G^{-5} \le T^{-\varepsilon}$ for $G \ge T^{1/5+\varepsilon}$. This means that we may replace, on the right-hand side of (4.5), $f(t,k)$ in the exponential by

$$-\tfrac{1}{4}\pi + 2\sqrt{2\pi kt} + \tfrac{1}{6}\sqrt{2\pi^3}k^{3/2}t^{-1/2}$$

times a series whose terms are of descending order of magnitude. The main contribution will thus come from the above term.

After this procedure we see that the integral in (4.5) bears close resemblance to the integral of the fourth moment of $E^*(t)$. The term $k^{3/2}t^{-1/2}$ in the exponential is treated by the use of Lemma 5, similarly as was done in the case of $\Sigma_4(X,N;u)$ in Section 3. In our case, due to the fact that $K \ge T^{1/3}$ may be assumed, there will be no sum corresponding to $\Sigma_3(X;u)$. Now we proceed similarly as in the proof of Theorem 1. We shall apply Lemma 5 as in the proof of Theorem 1. Developing the fourth power in (4.5) and performing a large number of integrations by parts, we see that only the values for which

$$|E| \;\le\; T^{\varepsilon-1/2}, \quad E = \sqrt{8\pi}(\sqrt{m}+\sqrt{n}-\sqrt{k}-\sqrt{l}\,)$$

will be relevant, where m,n,k,l are integers from $[K,\,K']$. Thus, by Lemma 1 (with $\delta = T^{-1/2+\varepsilon}K^{-1/2}$) and trivial estimation, their contribution to I_K will be

$$\begin{aligned}&\ll_\varepsilon T^{1+\varepsilon}K^{-1}(K^4T^{-1/2}K^{-1/2}+K^2)\\ &\ll_\varepsilon T^{1+\varepsilon}K^{5/2}T^{-1/2} \ll_\varepsilon T^{3+\varepsilon}G^{-5}.\end{aligned}$$

This yields the bound

$$\sum_{r=1}^{R}\left(\int_{t_r-G}^{t_r+G}|\zeta(\tfrac{1}{2}+it)|^2\,\mathrm{d}t\right)^4 \ll_\varepsilon RG^4T^\varepsilon + G^3T^{-1}T^{3+\varepsilon}G^{-5},$$

which is (1.17).

It remains to show how (1.17) gives the twelfth moment estimate (1.18). Write

$$\int_T^{2T}|\zeta(\tfrac{1}{2}+it)|^{12}\,\mathrm{d}t \le \sum_{r\le T+1}|\zeta(\tfrac{1}{2}+i\tau_r^*)|^{12}, \tag{4.6}$$

where for $r = 1,2,\dots$ we set

$$|\zeta(\tfrac{1}{2}+i\tau_r^*)| \;:=\; \max_{T+r-1\le t\le T+r}|\zeta(\tfrac{1}{2}+it)|.$$

Let $\{t_{r,V}\}$ be the subset of $\{\tau_r^*\}$ such that

$$V \le |\zeta(\tfrac12 + it_{r,V})| \le 2V \qquad (r = 1, \dots, R_V),$$

where clearly V may be restricted to $O(\log T)$ values of the form 2^m such that $\log T \le V \le T^{1/6}$, since $\zeta(\frac12 + it) = o(t^{1/6})$ (see [7, Chapter 7]). Now since we have (see e.g., [8, Theorem 1.2]), for fixed $k \in \mathbb{N}$,

$$|\zeta(\tfrac12 + it)|^k \ll \log t \int_{t-\frac12}^{t+\frac12} |\zeta(\tfrac12 + it + iu)|^k \,\mathrm{d}u + 1,$$

it follows that, for some points t_r' ($\in [T, 2T]$) with $r = 1, \dots, R'$, $R' \le R_V$, $1 \ll G \ll T$, $t_{r+1}' - t_r' \ge 5G$,

$$\begin{aligned}
R_V V^2 &\le \sum_{r=1}^{R_V} |\zeta(\tfrac12 + it_{r,V})|^2 \\
&\ll \sum_{r=1}^{R_V} \log T\Bigl(\int_{t_{r,V}-\frac12}^{t_{r,V}+\frac12} |\zeta(\tfrac12 + it)|^2 \,\mathrm{d}t + 1\Bigr) \\
&\ll \sum_{r=1}^{R'} \log T\Bigl(\int_{t_r'-G}^{t_r'+G} |\zeta(\tfrac12 + it)|^2 \,\mathrm{d}t\Bigr) + R_V \log T \\
&\le \log T (R')^{3/4}\Bigl(\sum_{r=1}^{R'} \int_{t_r'-G}^{t_r'+G} |\zeta(\tfrac12 + it)|^2 \,\mathrm{d}t\Bigr)^{1/4} + R_V \log T \\
&\ll_\varepsilon T^\varepsilon (R_V G + R_V^{3/4} T^{1/2} G^{-1/2}),
\end{aligned}$$

where the estimate of Theorem 2 was used, with R_V replacing R. If we take $G = V^2 T^{-2\varepsilon}$, then we obtain

$$R_V^{1/4} \ll_\varepsilon T^{1/2+\varepsilon} G^{-3/2},$$

which gives

$$R_V \ll_\varepsilon T^{2+\varepsilon} G^{-6} \ll_\varepsilon T^{2+\varepsilon} V^{-12}.$$

Then the portion of the sum in (4.6) for which $|\zeta(\frac12 + i\tau_r*)| \ge T^{1/10+\varepsilon}$ is

$$\ll \log T \max_{V \ge T^{1/10+\varepsilon}} R_V V^{12} \ll_\varepsilon T^{2+\varepsilon},$$

But for values of V such that $V \le T^{1/10+\varepsilon}$, the above bound easily follows from the large values estimate (the fourth moment) $R \ll_\varepsilon T^{1+\varepsilon} V^{-4}$. This shows that the

integral in (4.6) is $\ll_\varepsilon T^{2+\varepsilon}$, and proves (1.18). Note that the author [9, Corollary 1] proved the bound

$$\sum_{r=1}^{R}\Bigl(\sum_{t_r-G}^{t_r+G}|\zeta(\tfrac{1}{2}+it)|^4\,\mathrm{d}t\Bigr)^2 \ll RG^2\log^8 T + T^2G^{-1}\log^C T \tag{4.7}$$

for some $C > 0$, where $T < t_1 < \ldots < t_R \le 2T$, $t_{r+1}-t_r \ge 5G$ for $r = 1,\ldots,R-1$ and $1 \ll G \ll T$. The bound (4.7), which is independent of Theorem 2, was proved by a method different from the one used in this work. Like (1.17), the bound (4.7) also leads to the twelfth moment estimate (1.18).

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